

A New Bound for the Uniform Admissibility Theorem

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Let G be a reductive group defined over a p -adic field F , that is $G = F$ -points of a reductive group defined over F . A representation (π, V) of G is called smooth if for every $v \in V$ there exists an open compact subgroup $K < G$ such that $\pi(g)v = v$ for every $g \in K$. Let us denote by V^K the subspace of all K -fixed vectors in V . A representation is called admissible if V^K is finite-dimensional for every open compact subgroup $K < G$. In this work we consider only smooth admissible representations of G . Also, algebras are associative and with unit. Let K be a fixed open compact subgroup of G . In [Ber], Bernstein proved the following

Theorem 1 (Uniform Admissibility Theorem). *There exists a uniform constant $N(G, K)$ such that for every irreducible representation (π, V) of G there is an inequality $\dim V^K \leq N(G, K)$.*

In this work we reprove one of the two lemmas (Lemma 3) in Bernstein's original proof. As a result we obtain a sharper estimate for the bound $N(G, K)$ in the theorem. For the convenience of the reader we sketch here (almost without proofs) the main steps in Bernstein's proof. We will demonstrate the new bound on $\dim V^K$ only in the case $G = \mathrm{GL}_n(F)$. The case of a general reductive group is similar.

Let us reformulate the theorem in terms of Hecke algebras. Denote by $H(G, K)$ the Hecke algebra consisting of all compactly supported functions $f : G \rightarrow \mathbb{C}$ such that $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k \in K$. Denote by $H(G)$ the Hecke algebra consisting of all locally constant and compactly supported functions $f : G \rightarrow \mathbb{C}$. Let (π, V) be an irreducible representation of G such that $V^K \neq 0$. The Hecke algebra $H(G)$ acts on (π, V) . Note that the space V^K is an irreducible representation of $H(G, K)$. Indeed, let $0 \neq v \in V^K$, $0 \neq w \in V^K$. The representation (π, V) is irreducible, hence there exists a function $f \in H(G)$ such that $\pi(f)v = w$. Obviously $\pi(1_K * f * 1_K)v = w$ and the convolution $1_K * f * 1_K \in H(G, K)$. Thus we can reformulate Theorem 1 as the following

Theorem 2. *There exists a uniform constant $N(G, K)$ such that for every irreducible finite-dimensional $H(G, K)$ -module W there is an inequality $\dim W \leq N(G, K)$.*

The theorem follows from the following lemmas.

Lemma 3. [Ber, Proposition 2] *Let A_1, A_2, \dots, A_l be a commuting family of matrices in $M_{n \times n}(\mathbb{C})$. Let \mathcal{F} be the algebra of matrices in $M_{n \times n}(\mathbb{C})$ generated by A_1, A_2, \dots, A_l and the identity. Then*

$$\dim \mathcal{F} \leq (l+1)n^{2-\frac{2}{l+1}}.$$

We will prove this lemma later.

Lemma 4. [Ber, Proposition 1] *Let \mathcal{L} be an algebra over \mathbb{C} , \mathcal{A}, \mathcal{Z} commutative subalgebras in \mathcal{L} , $A_1, A_2, \dots, A_l \in \mathcal{A}$, $X_1, \dots, X_p, Y_1, \dots, Y_q \in \mathcal{L}$. Let us assume that \mathcal{Z} lies in the center of the algebra \mathcal{L} , $\mathcal{Z} \subset \mathcal{A}$, \mathcal{A} is the commutative algebra generated by A_1, \dots, A_l and \mathcal{Z} , and that any element $X \in \mathcal{L}$ can be written in the form $X = \sum X_i P_{ij} Y_j$, where $P_{ij} \in \mathcal{A}$, $(i = 1, \dots, p; j = 1, \dots, q)$. Then any irreducible finite dimensional representation of the algebra \mathcal{L} has dimension at most $(pq)^{(l+1)/2} (l+1)^{(l+1)/2}$.*

Proof. If $\rho : \mathcal{L} \rightarrow \mathrm{End}(V)$ is an irreducible representation, $\dim V = n$, then $\rho(\mathcal{Z}) = \mathbb{C} \cdot 1$ (by Schur's Lemma), $\dim \rho(\mathcal{L}) = n^2$ (by Burnside's theorem). By the conditions of Lemma 4,

$$\dim \rho(\mathcal{L}) \leq pq \dim \rho(\mathcal{A}) \leq pq (l+1)n^{2-2/(l+1)}.$$

Hence $n^2 \leq pq(l+1)n^{2-2/(l+1)}$, that is $n \leq (pq)^{(l+1)/2} (l+1)^{(l+1)/2}$. \square

Lemma 5. *The Hecke algebra $H(G, K)$ satisfies the conditions of Lemma 4.*

For the proof see [Ber, Lemma]. We only note that l is the F -semi-simple rank of the group G and the numbers p, q depend linearly on $[K_0 : K]$ where K_0 is a maximal compact subgroup of G such that $K \subset K_0$. Let us demonstrate the choices of p, q, x_i , and y_j in the case $G = \mathrm{GL}_n(F)$. Let $\mathcal{O} = \{x \in F \mid |x| \leq 1\}$ and let ϖ be a

generator of the maximal ideal in \mathcal{O} . Then $K_0 = GL_n(\mathcal{O})$ is a maximal compact subgroup of G . Let K be a “good enough” compact open subgroup of $GL_n(\mathcal{O}_F)$, for example a congruence subgroup of $GL_n(\mathcal{O}_F)$,

$$K = K_m := \{x \in G \mid \|1 - x\| \leq \|\varpi\|^m\},$$

where $m \geq 1$ and $\|x\| = \max |x_{ij}|$. In this case one can take a_j ($j = 1, \dots, n-1$) to be a diagonal matrix, $(a_j)_{ii} = 1$ for $i \leq j$ and $(a_j)_{ii} = \varpi$ for $i > j$ and define $A_j = 1_{K a_j K}$. Let \mathcal{Z} be the algebra generated by $1_{K g}$ for $g \in Z(G)$. Let \mathcal{A} be the algebra generated by \mathcal{Z} and A_j , $j = 1, \dots, n-1$. Let $p = q = [K_0 : K]$, decompose $K_0 = \cup K g_i$ $i = 1, \dots, p$ and choose $x_i = y_i = 1_{K g_i K}$. The elements $x_i, y_j, i, j = 1, \dots, p$ and the algebras \mathcal{A} , \mathcal{Z} , and $H(G, K)$ satisfy the conditions of Lemma 4. As a consequence, we obtain

$$\dim V^K \leq [GL_n(\mathcal{O}_F) : K]^n \cdot n^{n/2}$$

for every irreducible representation (π, V) of $GL_n(F)$. This improves Bernstein’s bound

$$\dim V^K \leq [GL_n(\mathcal{O}_F) : K]^{2^{n-1}}.$$

See [Ber] for more details. In the proof of Lemma 3 we need the following

Lemma 6. *Let $A \in M_{n \times n}(\mathbb{C})$ be a nilpotent matrix and let $m \geq 1$. Then*

$$\dim (\text{Span} \{A^m B \mid AB = BA\}) \leq \frac{n^2}{m}.$$

Proof. Suppose $A = \text{diag}(J_{l_1}, J_{l_2}, \dots, J_{l_k})$ where J_i is a Jordan block of dimension $i \times i$. The assertion $AC = CA$ for $C_{n \times n} = (C_{ij})$ with C_{ij} a block of dimension $l_i \times l_k$ means

$$J_{l_i} C_{ij} = C_{ij} J_{l_j}.$$

The dimension spanned by such blocks is $\min(l_i, l_j)$. Let us call this dimension d_{ij} and note that

$$d_{ij} \leq \frac{l_i l_j}{\max(l_i, l_j)}.$$

The matrix A^m kills every Jordan cell of size $\leq m$. Thus, the dimension of the vector space spanned by the matrices of the form $A^m B$ less than

$$\sum_{\max(l_i, l_j) \geq m} d_{ij} \leq \sum \frac{l_i l_j}{m} = \frac{n^2}{m}.$$

□

Proof of Lemma 3. By a standard argument, we can assume that the matrices A_1, \dots, A_l are nilpotent. Namely, let us view the matrices A_1, \dots, A_l as operators on $V = \mathbb{C}^n$. Since the algebra \mathcal{F} is commutative, we can decompose the space V into the direct sum of \mathcal{F} -invariant subspaces V_j such that for every $A \in \mathcal{F}$ and every j , the eigenvalues of $A|_{V_j}$ coincide. We can restrict ourselves to the case $V = V_j$, and subtracting suitable constants from the operators A_i , we may assume that all the A_i are nilpotent.

Let $x < n^2$, we will choose it later. Divide matrices of the form $A_1^{j_1} A_2^{j_2} \dots A_l^{j_l}$ into two families. One with $j_i < x$ for every $1 \leq i \leq l$ and the other with at least one of the powers $j_i \geq x$. The first family consists of x^l matrices. Let us estimate the dimension of the subspace generated by the second family. Suppose $j_1 \geq x$. The number of linearly independent matrices of the form $A_1^x B$ where $A_1 B = B A_1$ is at most $\frac{n^2}{x}$ by Lemma 6. Thus $\dim \mathcal{F}$ is bounded by

$$f(x) := \frac{ln^2}{x} + x^l.$$

The minimum is achieved for $f'(x_0) = 0$, $x_0 = n^{2/(l+1)}$. We obtain

$$\dim \mathcal{F} \leq ln^{2-2/(l+1)} + n^{2l/(l+1)} = (l+1)n^{2-\frac{2}{l+1}}.$$

□

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References

[Ber] I. N. Bernstein, All reductive p -adic groups are tame, Functional Analysis and its Applications 8, No.2, 3-6 (1974).